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THE LIMITING STEADY ROTATIONS OF A BODY ON A STRING WITH A SUSPENSION POINT ON THE AXIS OF SYMMETRY†

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The steady motions of an axisymmetrical rigid body suspended from a fixed base by a weightless undeformable rod or a non-twisting inextensible string are investigated. The case when the rod is fastened to the body at a point situated on its axis of dynamic symmetry is considered. All types of limiting equilibrium configurations which are possible when there is an unlimited increase in the angular velocity of rotation of the system about the vertical are analysed. Domains in which each type of limiting regular precession and permanent rotation can exist are constructed in the space of dimensionless parameters, and the nature of their asymptotic behaviour when the angular velocity increases is determined. The limiting motions which are possible in the case of suspension on a rod and impossible in the case of suspension on a string are investigated. © 2002 Elsevier Science Ltd. All rights reserved.

Consider an axisymmetrical rigid body with centre of mass G suspended on a weightless absolutely rigid rod (or a non-twisting inextensible tight string) at a point O_2, situated on the axis of symmetry of the body (Fig. 1). The other end of the rod (the point O_1) is connected to a device which ensures that the system rotates about a vertical axis with velocity ω. The motion of a body on a string suspension has been considered in a large number of papers. The most complete treatment can be found in [1], which also gives the most detailed review of the literature on the subject.

It is well known that a body on a rod (or string) can perform steady rotations (permanent rotations or regular precessions) [1]. The equations of regular precession have the form‡

$$\begin{aligned} &\omega^2(l\sin\alpha + a\sin\theta)\cos\alpha - g\sin\alpha = 0 \\ &\omega^2[ma(l\sin\alpha + a\sin\theta) + (A - C)\sin\theta]\cos\theta - (mga + C\omega\Omega)\sin\theta = 0 \end{aligned} \tag{1}$$

where m is the mass of the body, A and C are the central equatorial and axial moments of inertia of the body, a is the distance O_2G, l is the length of the rod O_1O_2, Ω is the angular velocity of natural rotation of the body, and the product $\omega\Omega$ is positive for direct precession and negative for inverse precession. The configuration of the system is defined by the angles α and θ between the descending vertical and the vectors $\mathbf{O_1O_2}$ and $\mathbf{O_2G}$ respectively (as usual we choose the anticlockwise direction as the direction for reading the angles). The angle θ, which defines the position of the body, takes values on the rod $\alpha \in [0, \pi]$, while for suspension on a string $\alpha \in [0, \pi/2]$. The reduction in the range of values of α in the latter case is due to the fact that no stress Balancing a compression of a string can develop; such a compression would be necessary for a steady rotation with $\alpha > \pi/2$. Regarding this see, for example, [2].

Note that permanent rotations of an axisymmetrical body are a special case of regular precessions and are described by system (1) when $\Omega = 0$.

In the problem of the steady motions of a rigid body suspended on a string, the evolution of the motion as a function of the angular velocity of rotation of the system is the clearest and easiest to interpret. Incidentally, this approach is also the most natural for an experimental investigation of the motion of a body driven by a string [3]. Here the problem inevitably arises of the types of conical motion which

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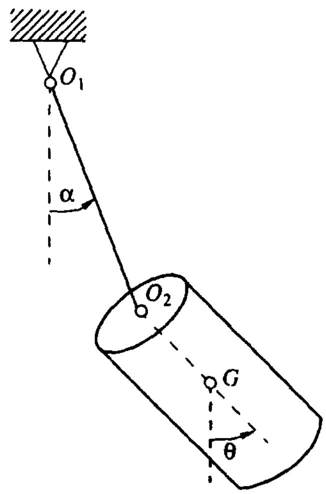

Fig. 1

exist for large values of ω. We will call such conical motions limiting motions. An analysis of limiting motions gives a fairly complete representation of the dynamic properties of the system and sometimes leads to the discovery of new interesting effects. Thus, the fact of the existence of limiting permanent rotations, for which one of the principal central axes of inertia of the body tends to coincide with the vertical, is the basis of the method of dynamic balancing of rapidly rotating bodies [4].

An investigation of limiting motions is extremely useful when it is not possible to make a complete investigation of the steady motions of a body on a string suspension. Such a situation arises, for example, in the problem of the steady rotations of a body fastened to a string at a point displaced from the principal central axis of inertia [5].

We will investigate all possible steady rotations of the mechanical system in question in the category of permanent rotations and regular precessions as $\omega \to \infty$, focusing our attention mainly on the domains in which they exist and the way they change when the system parameters change.

After changing to dimensionless parameters

$$\varkappa = \frac{a}{l}(>0), \quad \sigma = \frac{A-C}{mal}(\neq 0), \quad \nu = \frac{C\Omega}{ma\sqrt{lg}}, \quad \varepsilon = \frac{1}{\omega}\sqrt{\frac{g}{l}} \tag{2}$$

Eqs (1) take the form

$$\begin{gathered}(\sin\alpha + \varkappa\sin\theta)\cos\alpha - \varepsilon^2\sin\alpha = 0 \\ [\sin\alpha + (\varkappa + \sigma)\sin\theta]\cos\theta - \varepsilon(\varepsilon + \nu)\sin\theta = 0\end{gathered} \tag{3}$$

where, when considering rapid rotations, ε is a small parameter. We will seek a solution of system (3), i.e. the angles α and θ, characterizing steady motion, in the form of series in powers of ε

$$\alpha = \alpha_0 + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \ldots, \qquad \theta = \theta_0 + \varepsilon\theta_1 + \varepsilon^2\theta_2 + \ldots, \tag{4}$$

Substituting series (4) into Eqs (3) and retaining only terms of the zeroth order in ε, we obtain the system of equations

$$(\sin\alpha_0 + \varkappa\sin\theta_0)\cos\alpha_0 = 0, \qquad [\sin\alpha_0 + (\varkappa + \sigma)\sin\theta_0]\cos\theta_0 = 0 \tag{5}$$

which defines the limiting steady rotations of the body on the rod (string).

The following non-trivial solutions ($\sin\alpha_0 \neq 0$, $\sin\theta_0 \neq 0$) of system (5) exist

$$\begin{gathered}1)\ \alpha_0 = \pi/2,\ \ \theta_0 = \pi/2; \quad 2)\ \alpha_0 = \pi/2,\ \ \theta_0 = -\pi/2; \\ 3)\ \alpha_0 = \pi/2,\ \ \sin\theta_0 = -1/(\varkappa+\sigma); \quad 4)\ \sin\alpha_0 = \varkappa,\ \ \theta_0 = -\pi/2\end{gathered} \tag{6}$$

Solutions 1 and 2 exist for any system parameters and correspond to horizontal positions of the rod and the axis of symmetry of the body, where in case 1 the system is "unfolded" and in case 2 it is "folded".

Solutions (3) are only possible when $|\varkappa + \sigma| > 1$. In this case the rod is horizontal and the axis of symmetry of the body is inclined. We draw attention to the fact that Solution 3 when $\varkappa + \sigma < -1$ differs qualitatively from Solution 3 when $\varkappa + \sigma > 1$. In the first case

$$\theta_0^{(1)} = \theta_* = \arcsin(1/(\varkappa+\sigma|), \quad \theta_0^{(2)} = \pi - \theta_*$$

and the rod with the body are deflected from the vertical in the same direction. In the second case

$$\theta_0^{(1)} = -\theta_*, \quad \theta_0^{(2)} = -\pi + \theta_*$$

and the rod with the body are deflected in opposite directions. In [1] these motions were called "pendulum-like" and "regular bumpy" respectively.

Finally, Solutions 4, which exist when $\varkappa < 1$, correspond to an inclined rod ($\alpha_0^{(1)} = \arcsin \varkappa$, $\alpha_0^{(2)} = \pi - \arcsin \varkappa$) and a horizontal axis of symmetry of the body. Note that since the condition $l \sin \alpha_0 = a$ holds for Solutions 4, the centre of mass of the body is situated on the fixed vertical.

In order to investigate how the steady state evolves when it tends asymptotically to the corresponding limiting motion, we need to determine the signs of the first non-zero corrections to the limit values α_0 and θ_0. Substituting series (4) into Eqs (3), equating coefficients of terms of the first and second powers in ε and taking into account the obvious equality

$$\cos\alpha_0 \cos\theta_0 = 0 \tag{7}$$

which holds for all solutions (6), we obtain the following systems of equations

$$\begin{aligned} &\alpha_1(\cos 2\alpha_0 - \varkappa \sin\alpha_0 \sin\theta_0) = 0 \\ &\theta_1[(\varkappa+\sigma)\cos 2\theta_0 - \sin\alpha_0 \sin\theta_0] = \nu \sin\theta_0 \end{aligned} \tag{8}$$

$$\begin{aligned} &\alpha_2(\cos 2\alpha_0 - \varkappa\sin\alpha_0\sin\theta_0) = \sin\alpha_0 + \alpha_1^2\sin 2\alpha_0 + \\ &+\varkappa\left(\alpha_1\theta_1\sin\alpha_0\cos\theta_0 + \frac{\alpha_1^2+\theta_1^2}{2}\cos\alpha_0\sin\theta_0\right) \\ &\theta_2[(\varkappa+\sigma)\cos 2\theta_0 - \sin\alpha_0\sin\theta_0] = (\sin\theta_0 + \nu\theta_1\cos\theta_0) + (\varkappa+\sigma)\theta_1^2\sin 2\theta_0 + \\ &+\alpha_1\theta_1\cos\alpha_0\sin\theta_0 + \frac{\alpha_1^2+\theta_1^2}{2}\sin\alpha_0\cos\theta_0 \end{aligned} \tag{9}$$

The coefficients of the first-order corrections α_1 and θ_1 are found from system (8), where it can be seen from the first equation of (8) that for all limiting solutions (6) $\alpha_1 = 0$, with the exception of two degenerate cases. In the first case $\varkappa = 1$ and the correction α_1 becomes indeterminate for Solution 2. Here the motion is in fact similar to the motion of a rigid body with a fixed point (the centre of mass), since when $\alpha_0 = \pi/2$, $\theta_0 = -\pi/2$, $a = l$ the centre of mass G coincides with the point at which the string is attached to the fixed base O_1. In the second case α_1 becomes an indeterminate quantity for Solution 3 if $\sigma = 0$. Here the body degenerates into a sphere ($A = C$). These special cases will not be considered further.

From the second equation of (8) we find

$$\begin{aligned} &1)\ \theta_1 = -\frac{\nu}{\varkappa+\sigma+1}, \quad 2)\ \theta_1 = \frac{\nu}{\varkappa+\sigma-1} \\ &3)\ \theta_1 = \frac{\nu}{1-(\varkappa+\sigma)^2}, \quad 4)\ \theta_1 = \frac{\nu}{\sigma} \end{aligned} \tag{10}$$

Since the first-order corrections for the angle α_0 are equal to zero, from the first equation of (9) we obtain the following coefficients of the second-order corrections α_2

$$1)\ \alpha_2 = -\frac{1}{\varkappa+1},\quad 2)\ \alpha_2 = \frac{1}{\varkappa-1}$$

$$3)\ \alpha_2 = -\frac{\varkappa+\sigma}{\sigma},\quad 4)\ \alpha_2 = \frac{\varkappa}{1-\varkappa^2}\left(1-\frac{\nu^2}{2\sigma^2}\cos\alpha_0\right) \tag{11}$$

It follows from relations (10) that the sign of θ_1 changes on the straight line $\varkappa + \sigma = -1$ for Solution 1 and on the straight line $\varkappa + \sigma = 1$ for Solution 2. For Solution 4 the coefficient θ_1 changes sign when $\sigma = 0$, i.e. when the body degenerates into a sphere. Moreover, the sign of θ_1 for all solutions (6) depends on the sign of ν, i.e. on whether direct or inverse precession is considered.

It can be seen from relation (11) that $\alpha_2 < 2$ for Solution 1, α_2 changes sign on the straight line $\varkappa = 1$ for Solution 2, and the sign of the correction changes on the straight line $\sigma = 0$ for Solutions 3.

For Solutions 4 the sign of the correction depends on the type of solution. For the first type of solution $\alpha_0 = \alpha_0^{(1)}$ we have

$$\alpha_2 = \frac{\varkappa}{1-\varkappa^2}\left(1-\frac{\nu^2}{2\sigma^2}\sqrt{1-\varkappa^2}\right)$$

and consequently $\alpha_2 > 0$ when

$$\nu^2 < \nu_*^2 = \frac{2\sigma^2}{\sqrt{1-\varkappa^2}}$$

and $\alpha_2 < 0$ when $\nu^2 > \nu_*^2$. Hence we can distinguish slow and fast natural rotations depending on whether the first or second condition is satisfied. Hence, when $\alpha_0 = \alpha_0^{(1)}$ we have $\alpha_2 > 0$ for slow natural rotations and $\alpha < 0$ for fast ones. For the solution $\alpha_0 = \alpha_0^{(2)}$ the correction α_2 is always positive.

The results obtained are shown in Figs 2–5, where we show the domains in which different types of limiting solutions exist in the plane of the parameters $\varkappa$ and σ for the case of direct precession ($\nu > 0$), taking into account the nature of their asymptotic behaviour when $\omega \to \infty$, i.e. taking into account the signs of the first non-zero corrections α_2 and θ_1, in this case the equilibrium configurations of the system are represented schematically by the two sections O_1O_2 and O_2G. Note that in the case of inverse precession ($\nu < 0$) the direction in which the axis of the body tends towards its limiting position reverses.

No corresponding solutions exist in the hatched regions in Figs 4 and 5. Solution 3 (Fig. 4) changes into Solution 2 on the boundaries $\varkappa + \sigma = 1$, and into Solution 1 on the boundary $\varkappa + \sigma = -1$. Solution 4 (Fig. 5) changes into Solution 2 on the boundary $\varkappa = 1$.

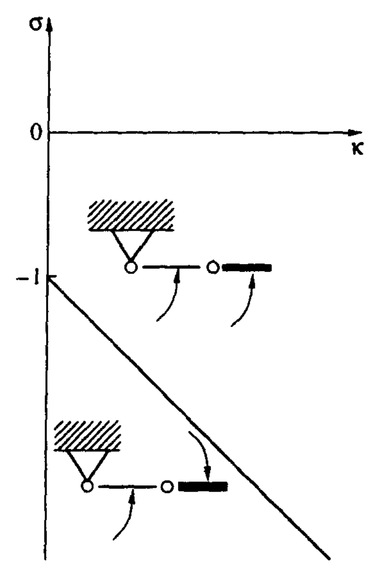

Fig. 2

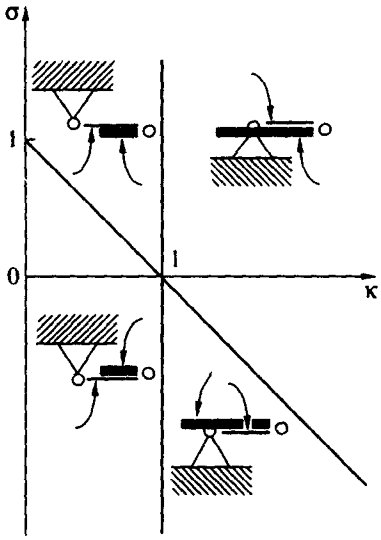

Fig. 3

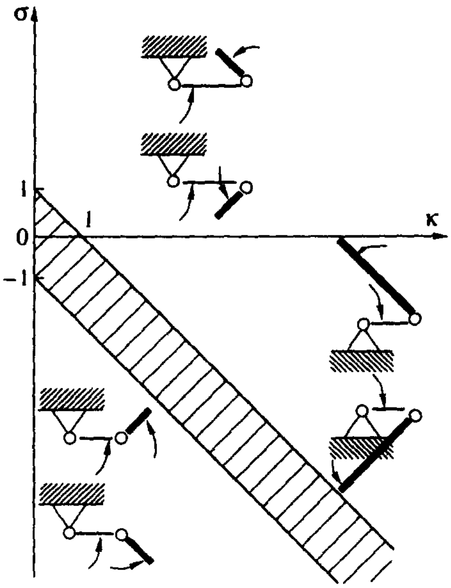

Fig. 4

As already mentioned, a feature of Solutions 4 is the fact that a different kind of asymptotic behaviour is possible for the case of slow and fast natural rotations. In Fig. 5, for those types of solutions for which this difference occurs, the direction in which the rod tends to its limiting position is shown by the continuous arrow when $\nu^2 < \nu_*^2$ and by the dashed arrow when $\nu^2 > \nu_*^2$.

We recall that, since there is no compression stress in the string, when the body is suspended on a string only those steady motions are possible for which the point where the string is attached to the body is situated below the point where the string is attached to the fixed base. This limitation does not apply when the body is suspended on a rod. Hence, all the steady motions of the system can be divided into two classes. Some of these are possible for both a rod and a string while the others are only possible for a rod. The same, naturally, applies to the limiting states. In Figs 2–5 the conventional representation of the suspension indicates that the corresponding type of steady motion is possible in both cases. The inverted suspension indicates that this case only occurs for a rod.

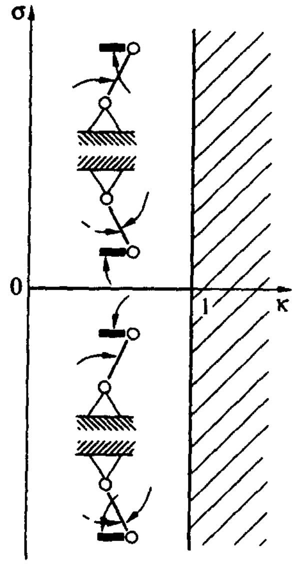

Fig. 5

As analysis of Figs 2–5 shows that the number of limiting regular precessions possible in a specific system depends only on the two dimensionless parameters $\varkappa$ and σ, and in the case of a rod the number is 2, 4 or 6. All six limiting solutions exist simultaneously if $|\varkappa + \sigma| > 1$, $\varkappa < 1$; when $|\varkappa + \sigma| < 1$, $\varkappa > 1$ there are only two such solutions; in the remaining part of the half-plane ($\varkappa > 0$, σ) there are four limiting solutions. In the case of a string suspension the number of limiting regular precessions is reduced. In the regions $|\varkappa + \sigma| > 1$, $\varkappa < 1$ five solutions exist simultaneously, in the region $\varkappa + \sigma > -1$, $\varkappa > 1$, $\sigma < 0$ only one solution is possible, while in the remaining part of the half-plane there are three solutions.

We will now consider permanent rotations as a special case of regular precessions when $\nu = 0$. Limiting solutions (6) obviously occur in this case also, but the expressions for the first non-zero corrections lack precision. The correction α_2 in expression (11) is only changed for Solution 4, i.e.

$$4)\ \alpha_2 = \frac{\varkappa}{1-\varkappa^2} \tag{12}$$

On the other hand, all the first-order corrections for the angle θ, as can be seen from (10), vanish when $\nu = 0$. Hence we must determine the second-order corrections θ_2. From the second equation of (9), using (7) and the fact that $\alpha_1 = \theta_1 = 0$, we obtain

$$1)\ \theta_2 = -\frac{1}{\varkappa+\sigma+1},\quad 2)\ \theta_2 = \frac{1}{\varkappa+\sigma-1},\quad 3)\ \theta_2 = \frac{1}{1-(\varkappa+\sigma)^2},\quad 4)\ \theta_2 = \frac{1}{\sigma} \tag{13}$$

The domains in which limiting permanent rotations can exist, their types and the nature of their evolution for Solutions 1, 3 and 4 are identical with the corresponding cases of direct regular precession and hence can be illustrated by Figs 2, 4 and 5. In this case, of course, the case of slow natural rotation in Fig. 5 corresponds to Solution 4.

Solution 2 possesses an additional feature, which consists of the fact that, when the angular velocity increases, the axis of dynamic symmetry of the body tends to coincide with the rod and the system is "folded". In this connection the question of whether the axis of symmetry of the body will lie above or below the rod as one approaches the limiting solution asymptotically is of some interest. Since, in this case, the first non-zero corrections to the angles α and θ are quantities of the same order, the answer depends not only on the signs of the corrections α_2 and θ_2, but also on the ratio of their magnitudes when these corrections are of the same sign. It follows from (11) and (13) that for Solution 2 the coefficients α_2 and θ_2 are simultaneously negative in the domain

$$\varkappa < 1, \qquad \varkappa + \sigma < 1 \tag{14}$$

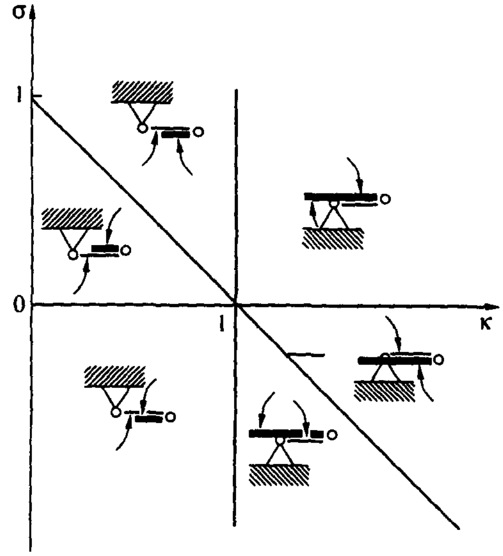

Fig. 6

and simultaneously positive in the domain

$$\varkappa > 1, \qquad \varkappa + \sigma > 1 \tag{15}$$

where α_2 if $\sigma = 0$.

Hence, the mutual position of the rod and the axis of dynamic symmetry of the body changes when the body degenerates into a sphere (Fig. 6). In domain (14) $0 > \alpha_2 > \theta_2$, if the body is dynamically prolate ($\sigma > 0$, i.e. $A > X$), and the body axis lies above the rod; if the body is dynamically oblate ($\sigma < 0$), then $0 > \theta_2 > \alpha_2$ and the body axis is below the rod. In domain (15), if $\sigma > 0$, then $\alpha_2 > \theta_2 > 0$ and the body axis is above the rod, while if $\sigma < 0$, then $\theta_2 > \alpha_2 > 0$ and the rod is above the body axis.

It is interesting to note that when the acceleration due to gravity g decreases, the parameter ε also becomes small. This indicates that all the results obtained when analysing the limiting permanent rotations (but not the regular precessions!) remain valid not only when $\omega \to \infty$, but also for a finite angular velocity if in this case $g \to 0$.

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